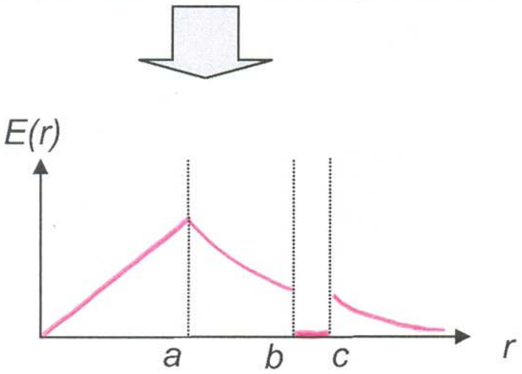
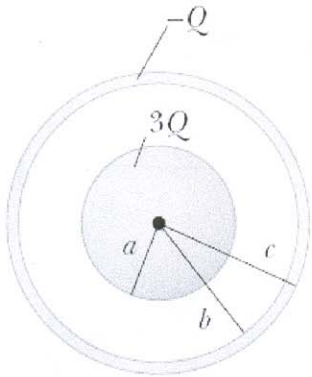
Q.1 *(25 points)* Gauss's Law

A solid insulating sphere of radius *a* carries a net positive charge *3Q*, uniformly distributed throughout its volume. Concentric with this sphere is a conducting spherical shell with inner radius *b* and outer radius *c*, and having a net charge -*Q*, as shown in the figure below.

(a) (4 pt.) Construct a spherical gaussian surface of radius $r > c$ and find the net charge enclosed by this surface.
(b) (4 pt.) What is the direction and magnitute of the electric field at $r > c$?
(c) (4 pt.) Determine the charge on the inner and outer surfaces of the conducting shell.
(d) (9 pt.) Find magnitute of electric field for $b<r<c$, $a<r<b$, and $r<a$.
(e) (4 pt.) Make a plot of the magnitude of the electric field versus *r*.





a) Net charge enclosed by $r>c$ Gaussian sphere $= 3Q - Q = 2Q$

b) From Gauss's Law: $\oint_{r>c} \vec{E}\cdot d\vec{A} = \frac{2Q}{\varepsilon_0} \Rightarrow E(r) = \frac{2Q}{4\pi\varepsilon_0}\frac{1}{r^2}$ $(r>c)$

Direction of $\vec{E}$: radially outward

c) Use the fact that electric field should vanish inside the conducting shell $(c>r>b)$. Therefore the net enclosed charge for $c>r>b$ must be zero.

Therefore @ inner surface: $q(r=b) = -3Q$

Since total charge on the shell should be $-Q$, the outer surface charge should be: $q(r=c) = +2Q$

d) For $b<r<c$, $E=0$

For $a<r<b$, $\oint \vec{E}\cdot d\vec{A} = \frac{+3Q}{\varepsilon_0}$, $E(r) = \frac{3Q}{4\pi\varepsilon_0}\frac{1}{r^2}$

For $r<a$, $\underbrace{\oint \vec{E}\cdot d\vec{A}}_{E4\pi r^2} = \frac{Q_{en}}{\varepsilon_0}$, $Q_{en} = 3Q\frac{r^3}{a^3}$, $E(r) = \frac{3Q}{4\pi\varepsilon_0}\frac{r}{a^3}$

Q.2 ***(25 points)*** **Electric Field and Potential**

The region $z<0$ is filled with a perfect conductor with some charge distribution. The region $z>0$ is the free space which has a potential profile given as $V(z>0)=V_0-Mz$, where V_0 and M are some known constants.

(a) (10 pt.) Determine the electric field vector everywhere ($z<0$ and $z>0$),

(b) (10 pt.) Sketch the electric field lines and equipotential surfaces,

(c) (5 pt.) What is the surface charge density on the conductor?

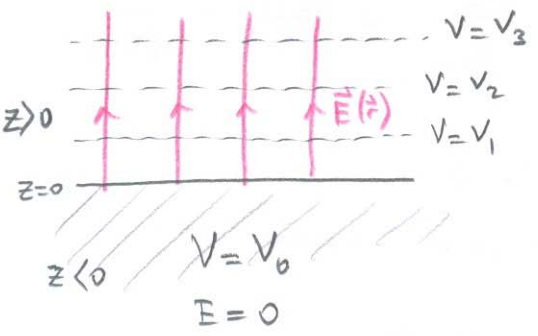
a) Using $V(\vec{r})$, we can determine $\vec{E}(\vec{r})$ by

$$\vec{E}(\vec{r}) = -\hat{i}\,\frac{\partial V(x,y,z)}{\partial x} - \hat{j}\,\frac{\partial V(x,y,z)}{\partial y} - \hat{k}\,\frac{\partial V(x,y,z)}{\partial z}$$

$$\vec{E}(\vec{r}) = -\hat{k}\,\frac{\partial V}{\partial z} = \hat{k}M \quad \text{for } z>0$$

$$\vec{E}(\vec{r}) = 0 \quad \text{for } z<0$$ ← inside the conductor, no E-field!

b)



equipotential surfaces are planes

c) Use $E_\perp = \dfrac{\sigma}{\varepsilon_0}$

$$\sigma = \varepsilon_0 M$$

Q.3 ***(25 points)*** **Cylindrical Capacitor**

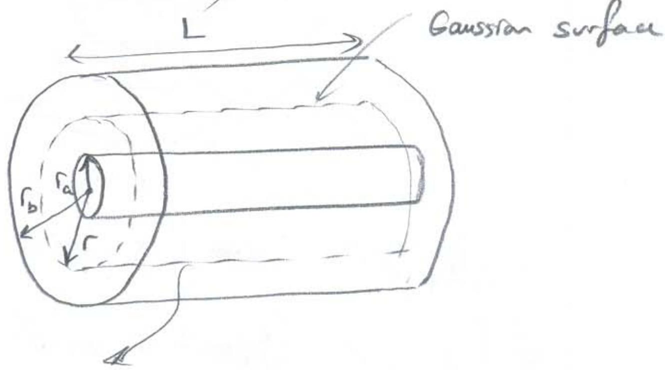
The inner cylinder of a cylindrical capacitor has radius r_a and linear charge density $+\lambda$. It is surrounded by a coaxial cylindrical conducting shell with inner radius r_b and linear charge density $-\lambda$.

(a) (12 pt.) Determine the electrostatic energy density in the region between the conductors at a distance r from the axis ($r_a < r < r_b$).

(b) (13 pt.) Determine the capacitance per unit length (you may or may not use the result for the previous part).

(This question was solved in the recitations)

a) $u = \frac{1}{2}\varepsilon_0 E^2$



Using Gauss's Law

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{en}}{\varepsilon_0} \quad ; \quad 2\pi r L E = \frac{\lambda L}{\varepsilon_0} \quad \text{for } r_a < r < r_b$$

$$\Rightarrow \quad E(r_a < r < r_b) = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Hence, $u = \dfrac{\lambda^2}{8\pi^2\varepsilon_0 r^2}$

b) Total stored energy is $U = \int u\, dV = 2\pi L \int u\, r\, dr$

$$U = \frac{L\lambda^2}{4\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{dr}{r} = \frac{\lambda^2 L}{4\pi\varepsilon_0}\ln(r_b/r_a)$$

$$\Rightarrow \quad \frac{U}{L} = \frac{\lambda^2 L^2}{4\pi\varepsilon_0}\ln\left(\frac{r_b}{r_a}\right) \quad ; \quad \text{From } \left(\frac{U}{L}\right) = \frac{Q^2}{2(C/L)} \; ;$$

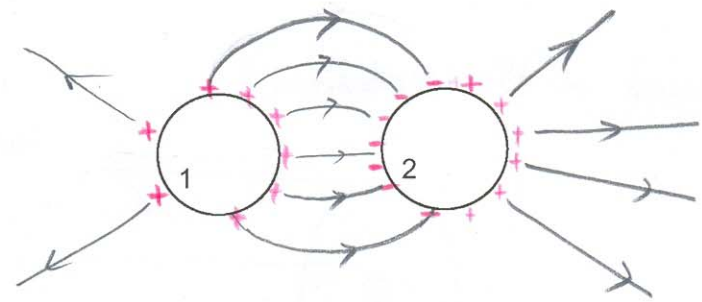
$\lambda^2 L^2 \rightarrow Q^2$

capacitance per unit length $\left\{ \dfrac{C}{L} = \dfrac{2\pi\varepsilon_0}{\ln(r_b/r_a)} \right.$

Q.4 *(25 points)* Conceptual Questions

Answer the following questions **in** the box for each part. To claim any credit for your discussions, **base your reasoning on physics (laws).**

(a) (8 pt.) A spherical conductor (labeled as 1) is positivley charged. Afterwards an initially uncharged spherical conductor (labeled as 2) is brought close to the first one. On the figure below, indicate the charge distribution on each conductor and the electric field lines. Also, state whether the net electric flux out of each conductor is negative, zero, or positive (with reasons).



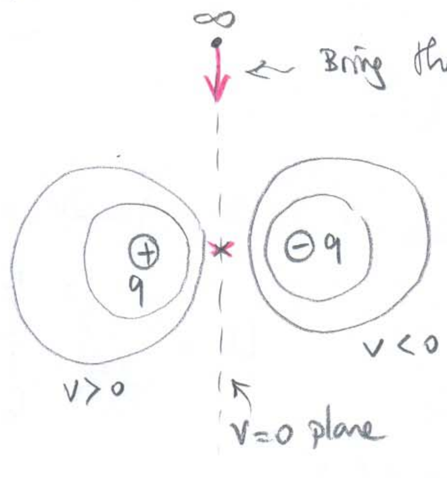
NB: Electric field lines should be ⊥ to conductor surfaces!

GAUSS' LAW
$\Phi_1 > 0$ (Sphere 1 is +ve'ly charged so net Electric flux is +ve)
$\Phi_2 = 0$ (Sphere 2 is overall uncharged, so net E-flux is zero)

(b) (5 pt.) The potential (relative to a point at infinity) midway between two charges of equal magnitude and opposite sign is zero. Is it possible to bring a test charge from infinity to this midpoint in such a way that no work is done in any part of the displacement? If so, describe the how it can be done. If it is not possible, explain why.

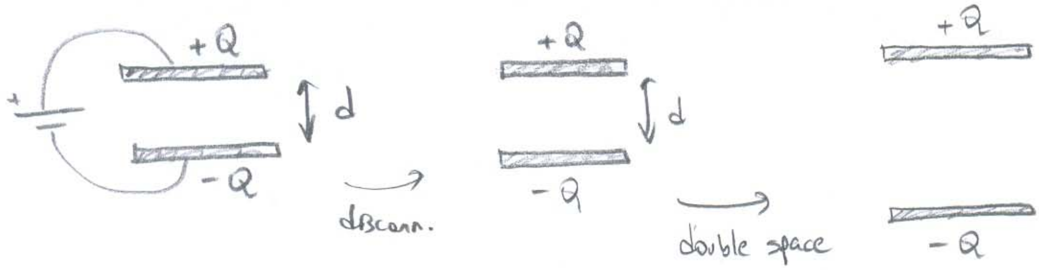
Yes, it is possible! This is an electric dipole problem.

Dipole Equipotential Surfaces



← Bring the test charge from infinity on this plane where $V \equiv 0$ so, no work will be done in any part.

(c) (6 pt.) A parallel-plate capacitor is charged by being connected to a battery and after charging it is disconnected from the battery. The separation between the plates is then doubled. What is the change in the (i) electric field, (ii) potential difference, (iii) stored energy with respect to initial configuration. Explain your reasoning.



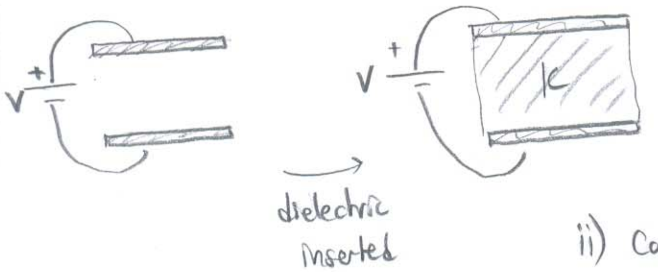
$C = \varepsilon_0 \frac{A}{d}$ $\qquad$ $C = \varepsilon_0 \frac{A}{d}$ $\qquad$ $C' = \varepsilon_0 \frac{A}{2d} = \frac{C}{2}$

$E = \frac{Q}{A}\varepsilon_0$; $V = dE$ $\qquad$ $E' = \frac{Q}{A}\varepsilon_0 = E$; $V' = 2V$

Based on these: $\qquad$ $U = \frac{1}{2}\frac{Q^2}{C}$ $\qquad$ $U' = \frac{1}{2}\frac{Q^2}{C'} = 2U$

i) E is same, ii) V is doubled, iii) U is doubled.

(d) (6 pt.) A parallel-plate capacitor is connected to a battery that maintains a fixed potential difference between the plates. If a slab of dielectric (of the same thickness with the plate separation) is then inserted between the plates, explain the change in the (i) the electric field, (ii) magnitude of the charge on the plates, (iii) stored energy with respect to initial configuration.



Battery is always connected, so V is same, hence

i) electric field is the same

ii) Capacitance is increased (by K times) due to the dielectric. Since $Q = CV$, with V kept constant the magnitude of charges on the plates should increase.

iii) $U = \frac{1}{2}CV^2$, V is const. but C is increased, so stored energy is also increased. (The polarization charges of the dielectric attract more charges from the battery).